SYSTEMATIC APPROACH FOR DISCRETE-TIME TO CONTINUOUS-TIME TRANSFORMATION OF $\Sigma\Delta$ MODULATORS

Hassan Aboushady and Marie-Minerve Louërat
University of Paris VI, LIP6/ASIM Laboratory
4, Place Jussieu, 75252 Paris Cedex 05, France
Email: Hassan.Aboushady@lip6.fr

***Abstract*—In this paper, we propose a systematic design method for continuous-time Sigma-Delta modulators. The modified-z-transform technique is used to take into account any variation occuring between two sampling instants in the continuous-time DAC feedback signal. The proposed discrete-time to continuous-time transformation method is general and well-suited to design automation. A fifth-order lowpass as well as a sixth order bandpass modulators with RZ and NRZ feedback signals are given as design examples.

I. Introduction

Continuous-Time (CT) $\Sigma\Delta$ modulators can operate at higher sampling frequency than their Discrete-Time (DT) counterparts. Furthermore, DT circuit techniques are suffering from high switch resistance in recent low-voltage CMOS processes. On the other hand, due to their mixed CT-DT nature, CT $\Sigma\Delta$ modulators are more difficult to design then DT modulators. Consequently, we start by designing a DT modulator and then we calculate its CT equivalent.

Fig.1 and Fig.2 show general forms of DT and CT $\Sigma\Delta$ modulators. The objective is to design the CT loop filter $H_c(s)$, for a given feedback DAC transfer function $H_{DAC}(s)$, so that the CT $\Sigma\Delta$ loop gain $G_c(z)$ is equal to the DT $\Sigma\Delta$ loop gain $G_d(z)$. This can be expressed by

$$\begin{aligned} G_d(z) &= G_c(z) \\ G_d(z) &= \mathcal{Z}\left[H_c(s)\;\; H_{DAC}(s)\right] \end{aligned} \tag{1}$$

Previous work on $\Sigma\Delta$ DT-CT equivalence [1]–[3], has usually solved equation (1) in the time domain using the following relation

$$\mathcal{Z}^{-1}\left[H_d(z)\right] = \mathcal{L}^{-1}\left[H_c(s)\;\; H_{DAC}(s)\right] \tag{2}$$

Due to the complicated mathematics involved in the computation of time-domain convolution, solving equation (2) is not well-suited for design automation and has usually been used for specific cases.

A more general transformation method, using state-space representation has been presented in [4]. Heavy use of matrix notation, singularity problems and the use of special control and optimization MATLAB functions [5] make the use this transformation technique rather difficult.

In this work, we propose to solve equation (1) directly in the z domain using the *modified-z-transform* technique [6]. While

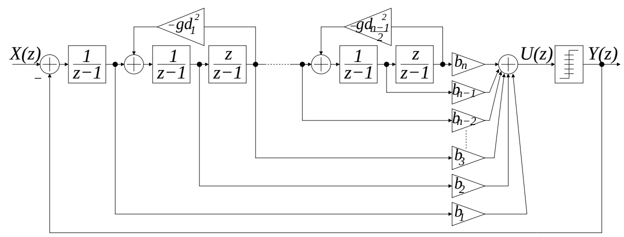

Fig. 1. General form of a discrete-time $\Sigma\Delta$ modulator.

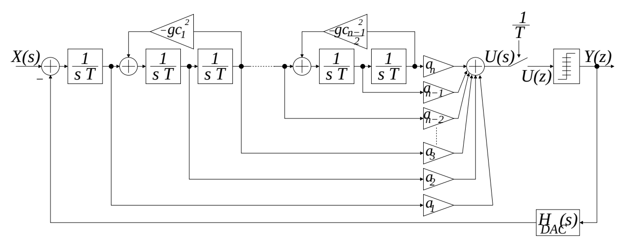

Fig. 2. General form of a continuous-time $\Sigma\Delta$ modulator.

avoiding the complex mathematics necessary to perform time-domain convolution, this technique enables us to get the z-transform of signals having variations between two sampling instants.

In this paper, we present a general method for an n^{th} order CT modulator. This method is valid for the different lowpass and bandpass $\Sigma\Delta$ topologies. The feedback DAC can be RZ or NRZ and the shape of the feedback signal can either be rectangular or non-rectangular.

II. z-Transform of Continuous-Time $\Sigma\Delta$ Loop Gain

A. Rectangular Feedback Signal

During a period T, the rectangular feedback signal, shown in Fig.3, can be described in the time domain by the following relationship:

$$h_{dac}(t) = u(t - t_d) - u(t - t_d - \tau) \tag{3}$$

where $u(t)$ is unit step function. Applying the *Laplace* transform, we get

$$H_{DAC}(s) = \frac{e^{-t_d s} - e^{-(t_d+\tau)s}}{s} \tag{4}$$

The z-transform of the CT $\Sigma\Delta$ loop gain can be expressed by

$$\mathcal{Z}\left[G_c(s)\right] \quad = \quad \mathcal{Z}\left[H_c(s)\;\; H_{DAC}(s)\right] \tag{5}$$

by substitution from equation (4) into equation (5), we have

$$G_c(z) = \mathcal{Z}\left[\frac{H_c(s)\ e^{-t_d s}}{s}\right] - \mathcal{Z}\left[\frac{H_c(s)\ e^{-(t_d+\tau)s}}{s}\right] \quad (6)$$

The conventional z-transform cannot be used to represent any variations occuring between two consecutive sampling instants. Since $0 \le t_d < T$ and $0 < t_d + \tau \le T$, another mathematical approach has to be used to represent equation (6) in the discrete-time z domain.
The modified-z-transform method is a modification of the z-transform method so that the output at any time between two consecutive sampling instants can be obtained [6].
Equation (6) is rewritten in the following form:

$$G_c(z) \quad = \quad \mathcal{Z}_{m_1}\left[\frac{H_c(s)}{s}\right] - \mathcal{Z}_{m_2}\left[\frac{H_c(s)}{s}\right] \quad (7)$$

where $m_1 = 1 - \frac{t_d}{T}$ and $m_2 = 1 - \frac{(t_d+\tau)}{T}$. Equation (7) is a general expression that can be used to obtain the loop gain $G_c(z)$ of CT $\Sigma\Delta$ with rectangular RZ or NRZ feedback signals. Now, in order to design a CT $\Sigma\Delta$ modulator which is equivalent to a well-known DT $\Sigma\Delta$ modulator, we use equations (7) and (1) to get the general expression for DT-CT equivalence.

Conversion tables from the *Laplace* domain to the z domain exists for the modified-z-transform as it is the case for the conventional z-transform [6]. Another method to calculate the modified-z-transform starting from the *Laplace* representation is the *Residue* theorem [7]. This method is systematic and much more convenient for design automation. Equation (7) can then be written in the following form

$$\begin{aligned} G_c(z) = & \sum_{p_i = \text{poles of } \frac{H_c(s)}{s}} \text{Residues of } \left.\frac{H_c(s)}{s}\frac{e^{m_1 T s}}{z - e^{Ts}}\right|_{\text{at } p_i} \\ - & \sum_{p_i = \text{poles of } \frac{H_c(s)}{s}} \text{Residues of } \left.\frac{H_c(s)}{s}\frac{e^{m_2 T s}}{z - e^{Ts}}\right|_{\text{at } p_i} \end{aligned} \quad (8)$$

Using equation (8), the loop gain of the CT $\Sigma\Delta$ $G_c(z)$ signal can be expressed in the DT z domain. Comparing the coefficients of the numerator and the denominator of $G_c(z)$ with those of the DT loop gain $G_d(z)$, we can deduce the coefficients of the CT loop filter $H_c(s)$. This will be explained in detail, for the different $\Sigma\Delta$ topologies, in the section III.
Note that equation (8) is a general relation valid for both RZ and NRZ rectangular feedback signal. In the special case of a NRZ feedback signal, with $t_d = 0$ and $\tau = T$, we have $m_1 = 1$ and $m_2 = 0$.

B. Non-Rectangular Feedback Signals

There are two main reasons which make it useful to be able to design CT $\Sigma\Delta$ modulators with non-rectangular feedback signals. First, it can be used to model non-idealities in the rectangular feedback, e.g. non-zero rise and fall time. The second reason is that it is theoretically possible to use different feedback shapes to reduce the modulator sensitivity to clock jitter noise.

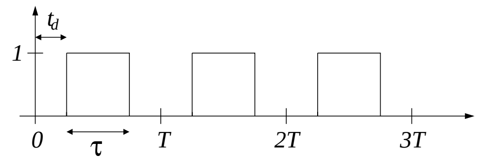

Fig. 3. Continuous-time rectangular feedback signal.

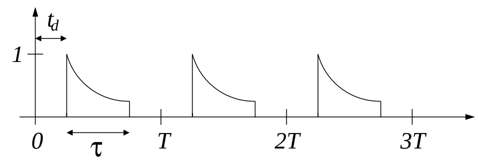

Fig. 4. Continuous-time decaying RC feedback signal.

In this section, we will show that the proposed CT-to-DT transformation method can be used for CT $\Sigma\Delta$ modulators with non-rectangular feedback signals. A decaying RC signal is taken as an example.

Following the same procedure described in section II.A., the decaying RC feedback signal, shown in Fig.4, is described in the time domain by the following relationship:

$$h_{dac_{rc}}(t) = e^{-\frac{1}{RC}(t-t_d)}[u(t-t_d) - u(t-t_d-\tau)] \quad (9)$$

Applying the *Laplace* transform, and then using the modified-z-transform technique, we can express the loop gain of a CT $\Sigma\Delta$ with a decaying RC feedback signal in the following form:

$$G_c(z) = \mathcal{Z}_{m1}\left[\frac{H_c(s)}{s + \frac{1}{RC}}\right] - e^{-\frac{\tau}{RC}}\mathcal{Z}_{m2}\left[\frac{H_c(s)}{s + \frac{1}{RC}}\right] \quad (10)$$

where $m_1 = 1 - \frac{t_d}{T}$ and $m_2 = 1 - \frac{(t_d+\tau)}{T}$. As described in section II.A., the modified-z-transform can be calculated using the residues method.

III. Systematic Design Approach

In this section, we will study, without loss of generality, 3 main topologies [5]:

- odd order Cascade of Resonators FeedForward form (CRFF), Fig.1 and Fig.2.
- even order CRFF, Fig.1 and Fig.2 without the first integrator.
- Cascade of Integrators FeedForward form (CIFF), Fig.1 and Fig.2 with $g_{d_i} = g_{c_i} = 0$.

The feedback forms CRFB and CIFB can directly be deduced from the feedforward forms and vice-versa.

The DT and CT loop gains of the CIFF, even CRFF and odd CRFF topology are listed in tables I, II and III respectively. DT loop gain calulations are straight forward. The modified-z-transform technique is used to get the CT loop gain.
Comparing the CIFF loop gain denominators of table I we find that the DT denominator coefficients $\beta_{d_n}, \ldots, \beta_{d_0}$ and the CT denominator coefficients $\beta_{c_n}, \ldots, \beta_{c_0}$ are identical. In fact,

TABLE I

Loop gain of CIFF (Fig.1 & 2 with $g_{d_i} = g_{c_i} = 0$).

n^{th} order CIFF		
DT	$G_d(z)$	$\frac{\alpha_{d_{n-1}} z^{n-1} + \alpha_{d_{n-2}} z^{n-2} + \dots + \alpha_{d_1} z + \alpha_{d_0}}{\beta_{d_n} z^n + \beta_{d_{n-1}} z^{n-1} + \dots + \beta_{d_1} z + \beta_{d_0}}$
CT	$G_c(z)$	$\frac{\alpha_{c_{n-1}} z^{n-1} + \alpha_{c_{n-2}} z^{n-2} + \dots + \alpha_{c_1} z + \alpha_{c_0}}{\beta_{c_n} z^n + \beta_{c_{n-1}} z^{n-1} + \dots + \beta_{c_1} z + \beta_{c_0}}$

TABLE II

Loop gain of even CRFF (Fig.1 & 2 without the 1^{st} integrator).

n^{th} order even CRFF		
DT	$G_d(z)$	$\frac{\alpha_{d_{n-1}} z^{n-1} + \alpha_{d_{n-2}} z^{n-2} + \dots + \alpha_{d_1} z + \alpha_{d_0}}{(z^2-(2-g_{d_1}{}^2)+1)\dots(z^2-(2-g_{d_{\frac{n}{2}}}{}^2)z+1)}$
CT	$G_c(z)$	$\frac{\alpha_{c_{n-1}} z^{n-1} + \alpha_{c_{n-2}} z^{n-2} + \dots + \alpha_{c_1} z + \alpha_{c_0}}{(z^2-2\ cos(g_{c_1})+1)\dots(z^2-2\ cos(g_{c_{\frac{n}{2}}})z+1)}$

TABLE III

Loop gain of odd CRFF (Fig.1 & 2).

n^{th} order odd CRFF		
DT	$G_d(z)$	$\frac{\alpha_{d_{n-1}} z^{n-1} + \alpha_{d_{n-2}} z^{n-2} + \dots + \alpha_{d_1} z + \alpha_{d_0}}{(z-1)(z^2-(2-g_{d_1}{}^2)+1)\dots(z^2-(2-g_{d_{\frac{n-1}{2}}}{}^2)z+1)}$
CT	$G_c(s)$	$\frac{\alpha_{c_{n-1}} z^{n-1} + \alpha_{c_{n-2}} z^{n-2} + \dots + \alpha_{c_1} z + \alpha_{c_0}}{(z-1)(z^2-2cos(g_{c_1})+1)\dots(z^2-2cos(g_{c_{\frac{n-1}{2}}})z+1)}$

these coefficients are independent from the loop filter coefficients $b_n, \dots, b_1$ and $a_n, \dots, a_1$. They are only dependent on the loop filter order, n. Comparing the CRFF loop gain denominators, the CT resonator feedback coefficient, g_{c_i}, can be described as a function of its DT counterpart, g_{d_i}, using the following relation:

$$g_{c_i} = cos^{-1}(1 - \frac{g_{d_i}{}^2}{2}) \tag{11}$$

Comparing the numerators of the DT and CT modulators loop gains described in tables I, II and III, we find that the DT numerator coefficients $\alpha_{d_{n-1}}, \dots, \alpha_{d_0}$ are function of the DT loop filter coefficients $b_n, \dots, b_1$ and the that CT numerator coefficients $\alpha_{c_{n-1}}, \dots, \alpha_{c_0}$ are function of the CT loop filter coefficients $a_n, \dots, a_1$ and the characteristics of the DAC feedback signals t_d and τ. For the CT modulator to be equivalent to the DT modulator, the following set of equations must be satisfied:

$$\begin{aligned} \alpha_{c_{n-1}}(a_1, \dots, a_n, t_d, \tau) &= \alpha_{d_{n-1}}(b_1, \dots, b_n) \\ \alpha_{c_{n-2}}(a_1, \dots, a_n, t_d, \tau) &= \alpha_{d_{n-2}}(b_1, \dots, b_n) \\ &\vdots \\ \alpha_{c_1}(a_1, \dots, a_n, t_d, \tau) &= \alpha_{d_1}(b_1, \dots, b_n) \\ \alpha_{c_0}(a_1, \dots, a_n, t_d, \tau) &= \alpha_{d_0}(b_1, \dots, b_n) \end{aligned} \tag{12}$$

Equation (12), can be written in the following form:

$$\begin{pmatrix} c_{11} & \dots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} d_{11} & \dots & d_{1n} \\ d_{21} & \dots & d_{2n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \dots & d_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

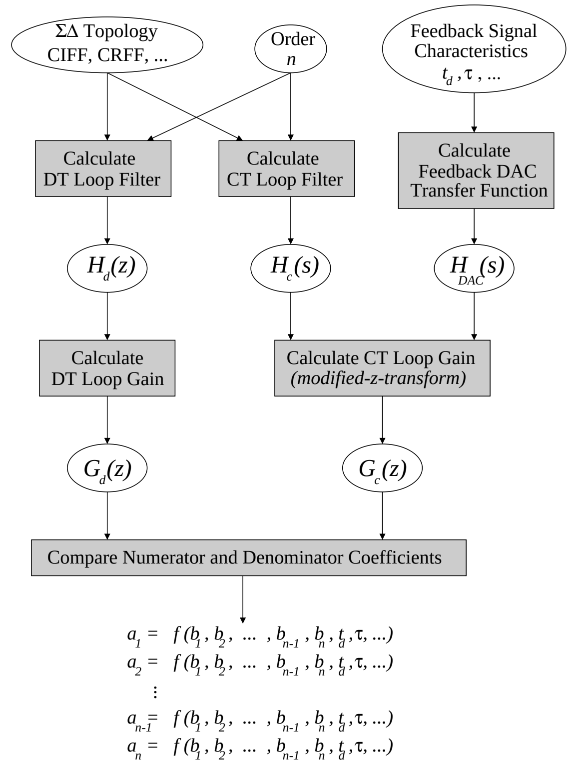

Fig. 5. Design procedure to find CT $\Sigma\Delta$ coefficients $(a_1, \dots, a_n)$ in function of DT $\Sigma\Delta$ coefficients $(b_1, \dots, b_n)$ and the feedback signal characteristics $(t_d, \tau, \dots)$.

where $c_{11}, \dots, c_{nn}$ are function of the DAC feedback signal characteristics t_d and τ. This equation can be expressed as

$$C \;\; A = D \;\; B \tag{14}$$

In order to get the CT vector coefficients A, we need to get the inverse of matrix C,

$$A = C^{-1} \;\; D \;\; B \tag{15}$$

The complete design procedure for RZ CT $\Sigma\Delta$ starting from a DT $\Sigma\Delta$, using the modified-z-transform technique, is summarized in Fig.5. This design procedure has been implemented in a symbolic mathematical tool *MAPLE* [8]. The mathematical tool has mainly been used to calculate the residues of equation (8) and the matrix inversion of equation (15). The advantage of using a symbolic tool is that the resulting CT coefficients $(a_1, \dots, a_n)$ are expressed in function of the DT coefficients $(b_1, \dots, b_n)$ and the feedback signal characteristics $(t_d, \tau, \dots)$. This means that for a given $\Sigma\Delta$ topology and order, the design procedure of Fig.5 is performed only once. Simple substitutions are then required to get the numerical values of the CT coefficients.

IV. Design Examples

The CT $\Sigma\Delta$ design technique, described in this paper, has been used by the authors in [9] and [10] to design a second and a third order CT modulators, respectively.

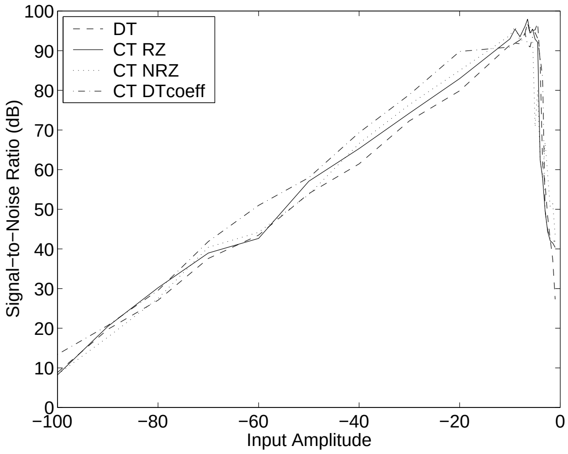

Fig. 6. Fifth order lowpass CRFF (OSR=64).

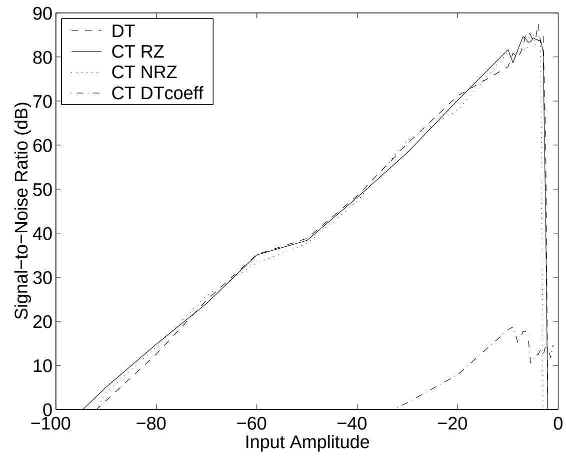

Fig. 7. Sixth order bandpass CRFF (OSR=64).

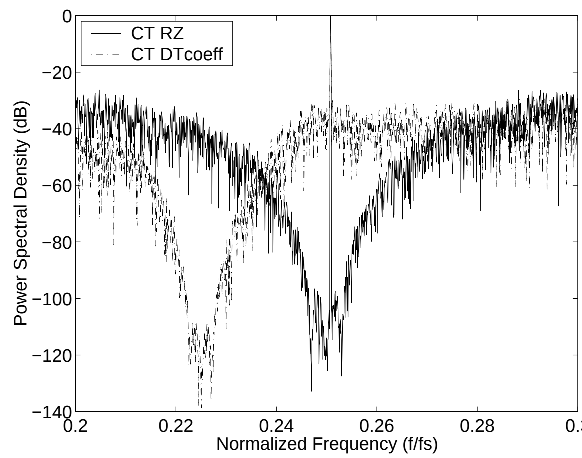

Fig. 8. Sixth order bandpass CRFF (input signal=-10dB, 16384 pts FFT).

Here we present a fifth order lowpass modulator and a sixth order bandpass modulator. The DT $\Sigma\Delta$ coefficients have been obtained using Richard Schreier's $\Sigma\Delta$ Toolbox [5]. Using the design procedure illustrated in Fig.5, the CT $\Sigma\Delta$ coefficients were obtained for both RZ and NRZ rectangular feedback signals. The RZ feedback signal had $t_d = \frac{1}{4}T$ and $\tau = \frac{3}{4}T$.

In order to study the behavior of the resulting CT $\Sigma\Delta$ modulator, several simulations have been performed to compare it with its DT counterpart. Fig.6 and Fig.7 show the Signal-to-Noise Ratio resulting from the simulation of the DT modulator (DT), the RZ CT modulator (CT RZ), the NRZ CT modulator (CT NRZ), and a RZ CT modulator simulated with the DT coefficients (CT DTcoeff). All the coefficients are scaled in order to obtain maximum swing at the output of the each integrator.

We can see in Fig.6, that in the case of the fifth order lowpass CRFF, there is very little difference between the performance of the DT modulator (DT), the calculated CT modulators (CT RZ and CT NRZ), and the CT modulator using the DT coefficients (CT DTcoeff). On the other hand, it is clear from Fig.7, that in the case of the sixth order bandpass CRFF, the CT modulator with DT coefficients has a very poor performance. Fig.8 shows the power spectral density of the sixth order $\Sigma\Delta$ of the calculated CT modulator and the CT modulator with the DT coefficients. It is obvious that the center frequency of the CT DTcoeff modulator is shifted from the required center frequency $\frac{f_s}{4}$.

V. Conclusion

A systematic DT-to-CT transformation method to design CT $\Sigma\Delta$ modulators has been presented. The method is general and can be used for rectangular and non-rectangular feedback signals with or without RZ. Implementing this method in a symbolic mathematical tool, has permitted us to design high-order lowpass and bandpass $\Sigma\Delta$ modulators with different topologies. It has been found that DT-to-CT transformation is particularly important in bandpass modulators.

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