Contents

List of Tables

List of Figures

Chapter 1

Cyclic Scheduling Problems for the Synthesis of Digital Signal Processing

Olivier Marchetti

Laboratoire LIP6, Université Paris 6 Pierre et Marie Curie

Alix Munier-Kordon

Laboratoire LIP6, Université Paris 6 Pierre et Marie Curie

Abstract This chapter is devoted to the study of Timed Weighted Event Graphs, which constitute a subclass of Petri Nets often considered for modelling embedded applications such as video encoders. Some basic recent mathematical properties are presented leading to algorithms checking the liveness and computing the optimum throughput of these systems.

1.1 Introduction

The design of embedded multi-media applications are nowadays a central industrial problem. These systems may often be modelled using Synchronous Dataflow Graphs (in short SDF) introduced by Lee and Messerschmitt [20, 21]. The vertices of these graphs correspond to programs. Each arc models a buffer used by the adjacent programs to communicate. It is briefly recalled in section 1.2.1 and is totally equivalent to a Weighted Event Graphs (in short WEG), which is a subclass of Petri Nets [29]. In this paper, we prefer WEG instead of SDF because of its importance in the computer scientists community.

There is an important literature on Timed (non Weighted) Event Graphs.

Indeed, as recalled in Section 1.3.2, they can be viewed as a subclass of uniform constraints as defined in Chapter [**?**]. As recalled in this chapter, for this class of graphs, the liveness (*i.e.* the existence of a schedule) and the computation of the optimum throughput are polynomial problems. The consequence is that most of the optimization problems on these structures are in $\mathcal{NP}$. For example, many authors developed efficient algorithms to solve the minimization of the number of tokens of an Event Graph for a given throughput [10, 14, 18] or some interesting variants [13].

For general Timed Weighted Event Graphs, there is no, to our knowledge, polynomial algorithm for the liveness or the computation of the maximum throughput despite original attempts to solve these problems [6]. Note that, for a slightly differents formalism called Computation Graph [16], Karp and Miller has shown that the deadlock existence problem is in $\mathcal{NP}$.

From a practical point of view, the optimization problems are solved using pseudo-polynomial algorithms to evaluate the liveness and the maximal throughput, which limits dramatically the size of tractable instances. In [30, 32], the liveness problem of a WEG is solved by using pseudo-polynomial algorithms mainly based upon a transformation of the WEG into an Event Graph called expansion introduced in [25, 26].

In [11], the authors consider model checking methods to evaluate the liveness of a Marked WEG. In the same way, they concluded in [12] that the computation of a state graph is not possible for large instances and that efficent methods are required to compute the optimal throughput.

In [5, 8, 28], the authors considered the computation of a periodic schedule for a Timed WEG with multi-objective functions such as the minimal schedule length and the minimal amount of memory. Several authors also consider the minimization of places capacities of a WEG for a given initial marking. This problem is $\mathcal{NP}$-complete even for Event Graphs as proved in [27] and several heuristics were developed to solve it [1, 2].

Surprisingly, some polynomial algorithms exists for optimization problems on Timed WEG with additional assumptions on the structure of the graph or the values of the throughput: a polynomial algorithm is developped in [22] to compute an initial live marking of a WEG that minimizes the places capacities. An approximation algorithm was also developed in [24] which maximizes the throughput for a place capacities at most twice from the minimum.

The aim of this paper is to present some basic mathematical properties on Timed WEG leading to polynomial algorithms to evaluate a sufficient condition of liveness and lower bounds of the optimum throughput. SDF and WEG are briefly presented in Section 1.2. Section 1.3 is devoted to the characterization of the precedence relations associated with a WEG and some basic technical important lemmas. Section 1.4 is dedicated to unitary graphs, which constitute an important subclass of WEG for the computation of the optimal throughput. Two important transformations of unitary WEG, namely the normalization and the expansions are detailed, with some other important properties. We present in Section 1.5 a polynomial algorithm to compute an optimal periodic schedule for a unitary WEG and leading to a sufficient condition of liveness and a lower bound to the optimum throughput. Section 1.6 is our conclusion.

1.2 Problem Formulation and basic notations

This section is devoted to the presentation of the problem. Synchronous Dataflow Graphs and Timed Weighted Event Graphs are briefly introduced. The two problems tackled here, namely the liveness and the determination of the maximal throughput of a Timed Marked Weighted Event Graph are then recalled.

1.2.1 Synchronous Dataflow Graphs

Synchronous Dataflow Graphs (SDF in short) are a formalism introduced and studied by Lee and Messerschmitt [20, 21] to model embedded applications defined by a set of programs $Pg_1, \ldots, Pg_n$ exchanging data using FIFO (First-In First-Out) queues. Each FIFO queue has exactly one input program Pg_i and one output program Pg_j and is modelled by an arc $e = (Pg_i, Pg_j)$ bi-valued by strictly positive integers val_i and val_j such that:

1. at the completion of one execution of Pg_i, $val_i(e)$ data are stored in the queue to be sent to Pg_j, and
2. at the beginning of one execution of Pg_j, $val_j(e)$ data are removed from the queue. If there are not enough data, Pg_j stops and waits for them.

This formalism suits particularly well for streaming applications such as video encoders and decoders which are nowadays crucial for economical reasons. Many real life examples of modelling such systems using SDF may be found in [4, 15]. Several complex environments for modelling and simulating these systems were also developed recently (*see* eamples [19, 33]). Figure 1.1 presents the modelling of a H263 decoder using a SDF presented in [31].

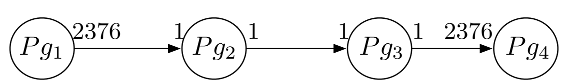

FIGURE 1.1: Modelling the H263 decoder using a SDF [31].

1.2.2 Timed Weighted Event Graphs

A Weighted Event Graph $\mathcal{G} = (T, P)$ (in short WEG) is given by a set of transitions $T = \{t_1, \ldots, t_n\}$ and a set of places $P = \{p_1, \ldots, p_m\}$. Every place $p \in P$ is defined between two transitions t_i and t_j and is denoted by $p = (t_i, t_j)$. Arcs (t_i, p) and (p, t_j) are valued by strictly positive integers denoted respectively by $u(p)$ and $v(p)$. At each firing of the transition t_i (*resp.* t_j), $u(p)$ (*resp.* $v(p)$) tokens are added to (*resp.* removed from) place p.

For every transition $t \in T$, $\mathcal{P}^+(t)$ (*resp.* $\mathcal{P}^-(t)$) denotes the set of places successors (*resp.* predecessors) of t in $\mathcal{G}$. More formally,

$$\mathcal{P}^+(t) = \{p \in P, \exists t' \in T/p = (t, t') \in P\} \text{ and}$$

$$\mathcal{P}^-(t) = \{p \in P, \exists t' \in T/p = (t', t) \in P\}.$$

For any integer $\nu > 0$ and any transition $t_i \in T$, $< t_i, \nu >$ denotes the νth firing of t_i.

An initial marking of the place $p \in P$ is usually denoted as $M_0(p)$ and corresponds to the initial number of tokens of p. A Marked Weighted Event Graph $G = (T, P, M_0)$ is a WEG with an initial marking. Figure 1.2 presents a marked place $p = (t_i, t_j)$.

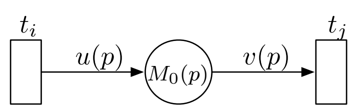

FIGURE 1.2: A marked place $p = (t_i, t_j)$.

A Timed Weighted Event Graph $\mathcal{G} = (T, P, M_0, \ell)$ is a Marked WEG such that each transition t has a processing time $\ell(t) \in \mathbb{N} - \{0\}$. Preemption is not allowed. The firing of a transition t at time μ requires then three steps:

1. if every place $p \in \mathcal{P}^-(t)$ has at least $v(p)$ tokens, then exactly $v(p)$ tokens are removed from p at time μ,
2. t is fired and is completed at time $\mu + \ell(t)$,
3. lastly, $u(p)$ tokens are placed in every place $p \in \mathcal{P}^+(t)$ at time $\mu + \ell(t)$.

A schedule σ is a function $S^\sigma : T \times \mathbb{N} - \{0\} \to \mathbb{Q}^+$ which associates, with any tuple $(t_i, k) \in T \times \mathbb{N} - \{0\}$, the starting time of the kth firing of t_i denoted by $S^\sigma_{<t_i,k>}$.

There is a strong relationship between a schedule σ and its corresponding instantaneous marking. Let $p = (t_i, t_j)$ be a place of P. For any value

$\mu \in \mathbb{Q}^+ - \{0\}$, let us denote by $E(\mu, t_i)$ the number of firings of t_i completed at time μ. More formally,

$$E(\mu, t_i) = \max\{q \in \mathbb{N}, S^\sigma_{<t_i,q>} + \ell(t_i) \leq \mu\}.$$

On the same way, $B(\mu, t_j)$ denotes the number of firings of t_j started up to time μ and

$$B(\mu, t_j) = \max\{q \in \mathbb{N}, S^\sigma_{<t_j,q>} \leq \mu\}.$$

Clearly,

$$M(\mu, p) = M(0, p) + u(p) \cdot E(\mu, t_i) - v(p) \cdot B(\mu, t_j).$$

The initial marking of a place $p \in P$ is usually denoted as $M_0(p)$ (*i.e.* $M_0(p) = M(0, p)$).

A schedule (and its corresponding marking) is feasible if $M(\mu, p) \geq 0$ for every tuple $(\mu, p) \in \mathbb{Q}^+ - \{0\} \times P$. The throughput of a transition t_i for a schedule σ is defined by

$$\tau_i^\sigma = \lim_{q \to +\infty} \frac{q}{S^\sigma_{<t_i,q>}}.$$

The throughput of σ is the smallest throughput of a transition: $\tau^\sigma = \min_{t_i \in T} \tau_i^\sigma$.

Throughout this paper, it is also assumed that transitions are non-reentrant, *i.e.* two successive firings of a same transition cannot overlap. This corresponds to

$$\forall t \in T, \forall q > 0, S^\sigma_{<t,q>} + \ell(t) \leq S^\sigma_{<t,q+1>}$$

Non-reentrance of a transition $t \in T$ can be modelled by a place $p = (t, t)$ with $u(p) = v(p) = 1$ and $M_0(p) = 1$. In order to simplify the figures, they are not pictured. However, most of the results presented here may be easily extended if some transitions are reentrant.

SDF and Timed WEG are equivalent formalisms: transitions may be associated to programs and places to FIFO queues. However, Timed Marked WEG simply models the data exchanged using tokens and is a sub-class of Petri Nets. We selected this last formalism in the rest of the paper.

1.2.3 Problem Formulation

Let us consider a given Timed Marked WEG $\mathcal{G} = (T, P, M_0, \ell)$. The two basic problems considered here are formally defined as follows:

Liveness: May every transition be fired infinitely often ?

$\mathcal{G}$ must be live since an embedded code has to be performed without interruption.

Maximal throughput: what is the maximal throughput of a feasible schedule ?

Remark that the earliest schedule (which consists in firing the transitions as soon as possible) always exists for live Marked Timed WEG and has a maximum throughput.

As recalled in Section 1.1, these two problems correspond to the checking stages [9] of most optimization problems on Timed Marked WEG. They must be efficiently solved to compute good solutions to most optimization problems on Timed WEG.

1.3 Precedence relations induced by a Timed Marked WEG

This section is dedicated to several basic technical properties on Timed Marked WEG. The precedence relations between the successive firings of two transitions adjacent to a place p are firstly characterized. Then, it is observed that for Event Graphs (*i.e.* $u(p) = v(p) = 1$ for every $p \in P$), these relations are uniform precedence constraints as defined in Chapter **??**. Some additional technical lemmas on precedence relations are lastly considered.

1.3.1 Characterization of the precedence relations

As defined in Section 1.2.2, a schedule σ is feasible if the number of tokens remains positive in every place. This constraint generates precedence relations between the firings of every couple of transitions adjacent to a place. Strict precedence relations between two firings is defined and characterized in the following.

DEFINITION 1.1 *Let a place* $p = (t_i, t_j)$ *and a couple of strictly positive integers* (ν_i, ν_j)*. There exists a (strict) precedence relation from* $< t_i, \nu_i >$ *to* $< t_j, \nu_j >$ *if*

Condition 1 $< t_j, \nu_j >$ *can be done after* $< t_i, \nu_i >$*;*

Condition 2 $< t_j, \nu_j - 1 >$ *can be done before* $< t_i, \nu_i >$ *while* $< t_j, \nu_j >$ *cannot.*

The following lemma characterized the couples of firings constrained by precedence relations:

LEMMA 1.1
A place $p = (t_i, t_j) \in P$ *with initially* $M_0(p)$ *tokens models a precedence*

relation between the ν_ith firing of t_i and the ν_jth firing of t_j iff

$$u(p) > M_0(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0).$$

PROOF By Definition 1.1, a place $p = (t_i, t_j) \in P$ with initially $M_0(p)$ tokens models a precedence relation from $< t_i, \nu_i >$ to $< t_j, \nu_j >$ iff Conditions 1 and 2 hold.

1. Condition 1 is equivalent to

$$M_0(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq 0.$$

2. Condition 2 is equivalent to

$$v(p) > M_0(p) + u(p) \cdot (\nu_i - 1) - v(p) \cdot (\nu_j - 1) \geq 0.$$

Combining these two inequalities, we obtain the inequality required. ∎

1.3.2 Timed Event Graphs

A Timed Event Graph is a WEG such that $u(p) = v(p) = 1$ for every place $p \in P$. The set of precedence relations induced by a Marked Timed Event Graph $\mathcal{G} = (T, P, M_0, \ell)$ can be modelled using uniform constraints as defined previously in Chapter **??**. Indeed, by Lemma 1.1, there exists a precedence relation between the ν_ith firing of t_i and the ν_jth firing of t_j induced by a place $p = (t_i, t_j)$ iff

$$1 > M_0(p) + \nu_i - \nu_j \geq 0,$$

which is equivalent to $\nu_j = \nu_i + M_0(p)$. A feasible schedule σ verifies then, for every place $p = (t_i, t_j) \in P$, the infinite set of precedence relations

$$\forall \nu > 0, S^{\sigma}_{<t_i,\nu>} + \ell(t_i) \leq S^{\sigma}_{<t_j,\nu+M_0(p)>}$$

which correspond exactly to a uniform precedence constraint $a = (t_i, t_j)$ with length $L(a) = \ell(t_i)$ and height $H(a) = M_0(p)$. So, liveness and computation of the maximal throughput may be both polynomially computed for this subclass of Timed Marked WEG using the algorithms recalled in Chapter **??**.

1.3.3 Equivalent places

For any place $p = (t_i, t_j) \in P$ with initially $M_0(p)$ tokens, $PR(p, M_0(p))$ denotes the infinite set of precedence relations between the firings of t_i and t_j induced by p.

DEFINITION 1.2 *Two marked places $p_1 = (t_i, t_j)$ and $p_2 = (t_i, t_j)$ are said equivalent if they induced the same set of precedence relations between the firings of t_i and t_j,* i.e. $PR(p_1, M_0(p_1)) = PR(p_2, M_0(p_2))$.

LEMMA 1.2
Two marked places $p_1 = (t_i, t_j)$ *and* $p_2 = (t_i, t_j)$ *with* $\frac{u(p_2)}{u(p_1)} = \frac{v(p_2)}{v(p_1)} = \frac{M_0(p_2)}{M_0(p_1)} = \Delta \in \mathbb{Q}^+ - \{0\}$ *are equivalent.*

PROOF Let us assume the existence of a precedence relation induced by p_1 between $< t_i, \nu_i >$ and $< t_j, \nu_j >$. Then, by Lemma 1.1, we get

$$u(p_1) > M_0(p_1) + u(p_1) \cdot \nu_i - v(p_1) \cdot \nu_j \geqslant \max(u(p_1) - v(p_1), 0).$$

$$\Updownarrow \times\Delta$$

$$\Delta.u(p_1) > \Delta.(M_0(p_1) + u(p_1) \cdot \nu_i - v(p_1) \cdot \nu_j) \geqslant \Delta.\max(u(p_1) - v(p_1), 0)$$

$$\Updownarrow$$

$$u(p_2) > M_0(p_2) + u(p_2) \cdot \nu_i - v(p_2) \cdot \nu_j \geqslant \max(u(p_2) - v(p_2), 0)$$

We conclude that p_2 induces a precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$, which completes the proof. $\square$

For every place $p \in P$, the greatest common divisor of the integers $u(p)$ and $v(p)$ is denoted by gcd_p, *i.e.* $gcd_p = gcd(u(p), v(p))$. The following lemma limits the possible values of the initial markings of a place to the multiples of gcd_p:

LEMMA 1.3
The initial marking $M_0(p)$ *of any place* $p = (t_i, t_j)$ *may be replaced by* $M_0^\star(p) = \left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p$ *tokens without any influence on the precedence relations induced by* p, i.e. $PR(p, M_0(p)) = PR(p, M_0^\star(p))$.

PROOF Using the Euclidean division of $M_0(p)$ by gcd_p, we get

$$M_0(p) = M_0^\star(p) + R_{gcd}(M_0(p)),$$

with $R_{gcd}(M_0(p)) \in \{0, \ldots, gcd_p - 1\}$.

$PR(p, M_0(p)) \subseteq PR(p, M_0^\star(p))$ Let us suppose that there exists a precedence relation from $PR(p, M_0(p))$ between $< t_i, \nu_i >$ and $< t_j, \nu_j >$. By Lemma 1.1,

$$u(p) > M_0(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0).$$

So, we get

$$u(p) > M_0^\star(p) + R_{gcd}(M_0(p)) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0).$$

Clearly,

$$u(p) > M_0^\star(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j.$$

Thus, since $M_0^\star(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j = 0 \mod (gcd_p)$, $\max(u(p) - v(p), 0) = 0 \mod (gcd_p)$ and $R_{gcd}(M_0(p)) \in \{0, \ldots, gcd_p - 1\}$, we get

$$M_0^\star(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0)$$

and the precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$ belongs also to $PR(p, M_0^\star(p))$.

$PR(p, M_0^\star(p)) \subseteq PR(p, M_0(p))$ Let us consider now a precedence relation from $PR(p, M_0^\star(p))$ between $< t_i, \nu_i >$ and $< t_j, \nu_j >$. By Lemma 1.1,

$$u(p) > M_0^\star(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0).$$

Clearly,

$$M_0^\star(p) + R_{gcd}(M_0(p)) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0).$$

Now, since $M_0^\star(p) + \nu_i \cdot u(p) - \nu_j \cdot v(p) = 0 \mod (gcd_p)$, we get

$$u(p) - gcd_p \geq M_0^\star(p) + u(p) \cdot \nu_i - v(p) \cdot \nu_j.$$

As $R_{gcd}(M_0(p)) < gcd_p$,

$$u(p) > M_0^\star(p) + R_{gcd}(M_0(p)) + u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0)$$

and the precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$ belongs also to $PR(p, M_0(p))$.

$\square$

In the rest of the paper, it is assumed that the initial marking of any place p is a multiple of gcd_p.

1.4 Unitary WEG

This section is dedicated to an important subclass of WEG called unitary graphs, which are firstly defined. We also recall briefly the interest of unitary graphs for checking the liveness or computing the optimal throughput of general Timed Marked WEG. The normalization of a WEG is then presented: it is a transformation introduced in [23] which simplifies the values of the marking functions of a unitary WEG. As recalled in Section 1.5, this transformation is the first step for the computation of an optimal periodic schedule. The expansion, which is another transformation presented in [26] is also detailed and the relationship between expansion and normalization is investigated. We present lastly a small example which illustrates the limit of the expansion for checking the liveness or computing the maximal throughput of a unitary WEG.

1.4.1 Definitions

A path μ of $\mathcal{G}$ is a sequence of k places such that $\mu = (p_1 = (t_1, t_2), p_2 = (t_2, t_3), \ldots, p_k = (t_k, t_{k+1}))$. If $t_{k+1} = t_1$ then μ is a circuit.

DEFINITION 1.3 *The weight (or gain) of a path μ of a WEG is the product* $\Gamma(\mu) = \prod_{p \in P \cap \mu} \frac{u(p)}{v(p)}$.

DEFINITION 1.4 *A strongly connected WEG $\mathcal{G}$ is unitary if every circuit c of $\mathcal{G}$ has a unit weight.*

Figure 1.3 presents a marked unitary WEG.

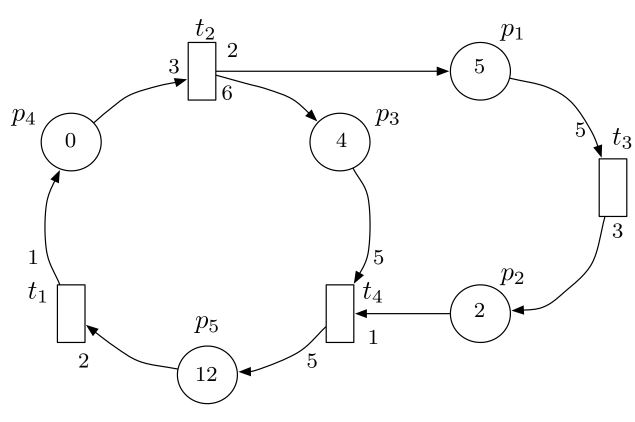

FIGURE 1.3: $\mathcal{G}$ is a Marked Unitary WEG.

1.4.1.1 Liveness and maximal throughput for general Timed Marked WEG

Let us consider in this subsection that $\mathcal{G}$ is a general Timed Marked WEG. Then several authors [16, 26, 32] proved that if $\mathcal{G}$ is live, the weight of every circuit c of $\mathcal{G}$ is at least 1. This condition is clearly not sufficient for the liveness: indeed, this condition is fulfilled for any usual Marked Event Graph with null markings, which is not live. However, this necessary condition of liveness allows to partition the transitions into unitary WEG called unitary components of $\mathcal{G}$ [26].

1. There are then two kinds of deadlocks in a general Timed Marked WEG:
   - If the circuit c causing the deadlock has a unit weight, it is included in a unitary component. This deadlock can be detected by studying the liveness of the corresponding unitary components of $\mathcal{G}$.
   - Otherwise, $\Gamma(c) > 1$. The only way known to detect it is by computing the firings of the transitions. Since $\Gamma(c) > 1$, this deadlock might occur quite quickly. However, there is no bound for the maximum number of firings needed to ensure the liveness of this class of circuits.
2. It is proved in [26] that the maximum throughput can be computed in polynomial time from the maximum throughput of each unitary component.

So, the study of the unitary graphs is fundamental to obtain efficient algorithms for both problems. Moreover, they corresponds to a wide class of interesting practical problems where the capacity of each place remains bounded [22].

1.4.2 Normalization of a unitary WEG

We present here the normalization of a unitary WEG. This transformation was originally presented in [23] and simplifies the marking functions.

DEFINITION 1.5 *A transition t_i is called normalized if there exists $Z_i \in \mathbb{N} - \{0\}$ such that $\forall p \in \mathcal{P}^+(t_i)$, $u(p) = Z_i$ and $\forall p \in \mathcal{P}^-(t_i)$, $v(p) = Z_i$. A unitary WEG $\mathcal{G}$ is normalized if all its transitions are normalized.*

By Lemma 1.2, functions $u(p)$, $v(p)$ and $M_0(p)$ of a place $p \in P$ can be multiply by any strictly positive integer without any influence on the precedence relations induced. Normalization of a Unitary WEG consists then in finding a vector $\gamma = (\gamma_1, \dots, \gamma_m) \in (\mathbb{N} - \{0\})^m$ such that

$$\forall t_i \in T, \forall (p_a, p_b) \in \mathcal{P}^+(t_i) \times \mathcal{P}^-(t_i), \gamma_a \cdot u(p_a) = \gamma_b \cdot v(p_b) = Z_i.$$

LEMMA 1.4
Let us suppose that $\mathcal{G}$ is a unitary WEG and (t_i, t_j) a couple of transitions from $\mathcal{G}$. Then, all paths from t_i to t_j have the same weight.

PROOF By contradiction, let us suppose that there exists two paths μ_1 and μ_2 from t_i to t_j with $\Gamma(\mu_1) \neq \Gamma(\mu_2)$. Let us denote by μ' a path from t_j to t_i. Then, circuits $\mu_1.\mu'$ and $\mu_2.\mu'$ verify

$$\Gamma(\mu_1.\mu') = \Gamma(\mu_1) \cdot \Gamma(\mu') \neq \Gamma(\mu_2) \cdot \Gamma(\mu') = \Gamma(\mu_2.\mu')$$

which is impossible since every circuit of $\mathcal{G}$ has a unit weight. ∎

THEOREM 1.1
Let $\mathcal{G}$ be a strongly connected WEG. $\mathcal{G}$ is normalizable iff $\mathcal{G}$ is unitary.

PROOF

$A \Rightarrow B$ Let us suppose that $\mathcal{G}$ is normalized. The weight of every circuit c is

$$\Gamma(c) = \prod_{p \in P \cap c} \frac{u(p)}{v(p)} = \left(\prod_{t_i \in T \cap c} Z_i \right) \cdot \left(\prod_{t_i \in T \cap c} \frac{1}{Z_i} \right) = 1$$

so $\mathcal{G}$ is unitary.

$B \Rightarrow A$ Let us suppose now that $\mathcal{G}$ is unitary. We must prove the existence of a vector $\gamma = (\gamma_1, \dots, \gamma_m) \in (\mathbb{N} - \{0\})^m$ such that

$$\forall t_i \in T, \forall (p_a, p_b) \in \mathcal{P}^+(t_i) \times \mathcal{P}^-(t_i), \gamma_a \cdot u(p_a) = \gamma_b \cdot v(p_b) = Z_i.$$

Let us build a directed valued graph $G = (P, E)$ as follows:

1. the set of vertices is the set of places,
2. $\forall t \in T$ and for every couple of places $(p_a, p_b) \in \mathcal{P}^+(t) \times \mathcal{P}^-(t)$, two arcs $e_1 = (p_a, p_b)$ and $e_2 = (p_b, p_a)$ are built with the respective values $y(e_1) = \frac{u(p_a)}{v(p_b)}$ and $y(e_2) = \frac{v(p_b)}{u(p_a)}$.

The problem consists then in finding a vector $\gamma = (\gamma_1, \dots, \gamma_m) \in (\mathbb{N} - \{0\})^m$ such that, for every arc $e = (p_a, p_b) \in E$, $\gamma_a \cdot y(e) \leq \gamma_b$.

From the proof of Bellman-Ford algorithm [7], γ exists iff every circuit c of G has a value $Y(c) = \prod_{e \in c} y(e) = 1$. Now,

$$Y(c) = \prod_{e \in c} y(e) = \prod_{p \in c} u(p) \cdot \prod_{p \in c} \frac{1}{v(p)} = \Gamma(c).$$

Since $\mathcal{G}$ is a unitary WEG, $\Gamma(c) = 1$ and thus $Y(c) = 1$, which completes the proof.

□

A polynomial algorithm to normalize a unitary WEG may be deduced from the proof of the last theorem. Indeed, a rational vector $\gamma^r = (\gamma_1^r, \ldots, \gamma_m^r)$ verifying the inequalities associated with the graph G can be build using Bellman-Ford algorithm [7]. An integer vector γ can be obtained from γ^r by setting $\gamma = A \cdot \gamma^r$, where A is the least common multiple of the denominators of the components of γ^r.

The associated system of the example pictured by Figure 1.3 is:

$$\begin{array}{l} Z_1 = 2\gamma_5 = \gamma_4 \\ Z_2 = 3\gamma_4 = 2\gamma_1 = 6\gamma_3 \\ Z_3 = 5\gamma_1 = 3\gamma_2 \\ Z_4 = 5\gamma_3 = \gamma_2 = 5\gamma_5 \end{array}$$

A minimum integer solution is $\gamma = (3, 5, 1, 2, 1)$ with $Z_1 = 2$, $Z_2 = 6$, $Z_3 = 15$ and $Z_4 = 5$. Figure 1.4 presents the corresponding normalized marked WEG.

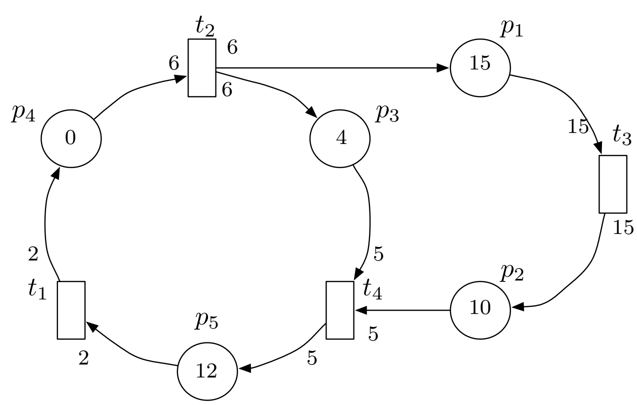

FIGURE 1.4: Normalized unitary graph of the marked WEG pictured by Figure 1.3.

1.4.3 Expansion of a unitary Timed Marked WEG

Let us suppose that $\mathcal{G}$ is a unitary Timed Marked WEG. The main idea here is to prove that the sets of precedence relations induced by $\mathcal{G}$ can be modelled using a Timed Event Graph.

1.4.3.1 Study of a place

We consider here a place $p = (t_i, t_j)$ of a WEG. Each transition t_i may be replaced by N_i transitions denoted by $t_i^1, \ldots, t_i^{N_i}$ such that for any $k \in \{1, \ldots, N_i\}$ and $r > 0$, the rth firing of t_i^k corresponds to the $\big((r-1)\cdot N_i + k\big)$th firing of t_i. Transitions $t_i^1, \ldots, t_i^{N_i}$ are called the *duplicates* of t_i.

Since transitions are supposed to be non-reentrant, these duplicates are included in a circuit as pictured by Figure 1.5.

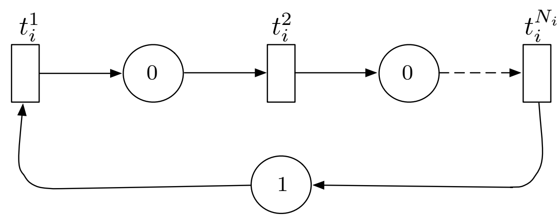

FIGURE 1.5: A circuit between N_i duplicates of t_i modelling the non-reentrant constraint.

LEMMA 1.5
Let $p = (t_i, t_j)$ be a place from a Timed Marked WEG. If p may be replaced by a finite set of non weighted places between the duplicates of t_i and t_j then the number of duplicates N_i and N_j of t_i and t_j must verify $\frac{N_i}{v(p)} = \frac{N_j}{u(p)}$.

PROOF Let us consider two positive integers ν_i and ν_j such that the inequality of Lemma 1.1 holds. It is assumed that the corresponding precedence relation is modelled by a place p_s between a duplicate of t_i and t_j. For any $r > 0$, the firings $\nu_i + r \cdot N_i$ and $\nu_j + r \cdot N_j$ are also constrained by a precedence relation induced by p_s, so

$$u(p) - M_0(p) > u(p)\cdot(\nu_i + r\cdot N_i) - v(p)\cdot(\nu_j + r\cdot N_j) \geq \max(u(p) - v(p), 0) - M_0(p).$$

These inequalities must be true for any value $r > 0$, so $N_i \cdot u(p) - N_j \cdot v(p) = 0$, which completes the proof. ∎

Conversely, let us suppose now that $\frac{N_i}{v(p)} = \frac{N_j}{u(p)} = s \in \mathbb{N} - \{0\}$. Two subcases are considered:

LEMMA 1.6
Let us suppose that $u(p) > v(p)$. If $\frac{N_i}{v(p)} = \frac{N_j}{u(p)} = s \in \mathbb{N} - \{0\}$ then p may be modelled by N_i non weighted places between the N_i and N_j duplicates of transitions t_i and t_j.

PROOF If $u(p) > v(p)$, the inequality of Lemma 1.1 becomes

$$\frac{M_0(p) + u(p) \cdot (\nu_i - 1)}{v(p)} < \nu_j \leq \frac{M_0(p) + u(p) \cdot (\nu_i - 1)}{v(p)} + 1$$

and thus $\nu_j = \left\lfloor \frac{M_0(p) + u(p) \cdot (\nu_i - 1)}{v(p)} \right\rfloor + 1$.

For every integer $\nu_i > 0$, r and s are two integers with $\nu_i = (r-1) \cdot N_i + s$, $r > 0$ and $s \in \{1, \ldots, N_i\}$. By definition of the duplicates, $< t_i, \nu_i >=< t_i^s, r >$.

We get $\nu_j = \left\lfloor \frac{M_0(p) + u(p) \cdot s}{v(p)} \right\rfloor + 1 + (r-1) \cdot N_j$. Let the sequences a_s and b_s such that $\left\lfloor \frac{M_0(p) + u(p) \cdot s}{v(p)} \right\rfloor + 1 = a_s \cdot N_j + b_s$ with $b_s \in \{1, \ldots, N_j\}$ then $\nu_j = (r - 1 + a_s) \cdot N_j + b_s$. We deduce that $< t_j, \nu_j >=< t_j^{b_s}, r + a_s >$.

Precedence relations between $< t_i^s, r >$ and $< t_j^{b_s}, r + a_r >$ with $s \in \{1, \ldots, N_i\}$ are modelled by a place $p'_s = (t_i^s, t_j^{b_s})$ with $M_0(p_s) = a_s$ tokens. ◻

For example, let us consider the place $p_5 = (t_4, t_1)$ of the Marked Unitary WEG pictured by Figure 1.3 and the number of duplicates $N_4 = 2$ and $N_1 = 5$. Sequences a_s and b_s, $s \in \{1, 2\}$ must verify $\left\lfloor \frac{12 + 5 \cdot s}{2} \right\rfloor + 1 = 5 \cdot a_s + b_s$ with $b_s \in \{1, \ldots, 5\}$. So we obtain the couples $(a_1, b_1) = (1, 4)$ and $(a_2, b_2) = (2, 2)$. Thus, p_5 may be replaced by the places $p'_1 = (t_4^1, t_1^4)$ with $M_0(p'_1) = 1$ and $p'_2 = (t_4^2, t_1^2)$ with $M_0(p'_2) = 2$.

LEMMA 1.7
Let us suppose now that $u(p) \| eqv(p)$. If $\frac{N_i}{v(p)} = \frac{N_j}{u(p)} = s \in \mathbb{N} - \{0\}$ then p may be modelled by N_j non weighted places between the N_i and N_j duplicates of transitions t_i and t_j.

PROOF If $u(p) \leq v(p)$, the inequality of Lemma 1.1 becomes

$$\frac{v(p) \cdot \nu_j - M_0(p)}{u(p)} + 1 > \nu_i \geq \frac{v(p) \cdot \nu_j - M_0(p)}{u(p)}$$

and thus $\nu_i = \left\lceil \frac{v(p) \cdot \nu_j - M_0(p)}{u(p)} \right\rceil$ with $\nu_j > \frac{M_0(p)}{v(p)}$.

For every integer $\nu_j > 0$, r and s can be defined as $\nu_j = (r-1) \cdot N_j + s$ with $r > 0$ and $s \in \{1, \ldots, N_j\}$. By definition of the duplicates, $< t_j, \nu_j >=< t_j^s, r >$.

We get $\nu_i = \left\lceil \frac{v(p) \cdot \nu_j - M_0(p)}{u(p)} \right\rceil = (r-1) \cdot N_i + \left\lceil \frac{s \cdot v(p) - M_0(p)}{u(p)} \right\rceil$. Let the sequences c_s and d_s such that $\left\lceil \frac{s \cdot v(p) - M_0(p)}{u(p)} \right\rceil = c_s \cdot N_i + d_s$ with $d_s \in \{1, \ldots, N_i\}$ then $\nu_i = (r-1+c_s) \cdot N_i + d_s$. We get $< t_i, \nu_i >=< t_i^{d_s}, r + c_s >$.

Remark that $\left\lceil \frac{s \cdot v(p) - M_0(p)}{u(p)} \right\rceil \leq \left\lceil \frac{N_j \cdot v(p) - M_0(p)}{u(p)} \right\rceil \leq N_i$. Thus $c_s \leq 0$ and the precedence relations between $< t_i^{d_s}, r + c_s >$ and $< t_j^s, r >$ for $k \in \{1, \ldots, N_j\}$ are modelled by a place $p_s =< t_i^{d_s}, t_j^s >$ with $M_0(p_s) = -c_k$ tokens. □

For example, let us consider the place $p_1 = (t_2, t_3)$ of the Marked Unitary WEG pictured by Figure 1.3 with $N_2 = 5$ and $N_3 = 2$. Sequences c_s and d_s verify $s \in \{1, 2\}$, $d_s \in \{1, \ldots, 5\}$ and $\left\lceil \frac{5 \cdot s - 5}{2} \right\rceil = 5 \cdot c_s + d_s$. So we obtain the couples $(c_1, d_1) = (-1, 5)$ and $(c_2, d_2) = (0, 3)$. Thus, p_1 may be replaced by the places $p'_1 = (t_2^5, t_3^1)$ with $M_0(p'_1) = 1$ and $p'_2 = (t_2^3, t_3^2)$ with $M_0(p'_2) = 0$.

1.4.3.2 Minimum expansion of a WEG

DEFINITION 1.6 *Let $\mathcal{G}$ be a strongly connected Marked WEG. $\mathcal{G}$ is expansible if there exists a vector $(N_1, \ldots, N_n) \in (\mathbb{N} - \{0\})^n$ such that every place $p = (t_i, t_j)$ can be replaced by non weighted places between the N_i and N_j duplicates of transitions t_i and t_j following Lemmas 1.6 and 1.7,* i.e. $\frac{N_i}{v(p)} = \frac{N_j}{u(p)}$.

An expansion of $\mathcal{G}$ is then a usual (non weighted) timed Event Graph which models exactly the same sets of precedence relations as $\mathcal{G}$. The number of duplicates of such a graph verifies the system $\Sigma(\mathcal{G})$ defined as:

$$\Sigma(\mathcal{G}): \quad \forall p = (t_i, t_j) \in P, \; \frac{N_i}{v(p)} = \frac{N_j}{u(p)} \in \mathbb{N} - \{0\}.$$

Let $\mathcal{S}$ be the set of solutions of $\Sigma(\mathcal{G})$.

THEOREM 1.2
Let $\mathcal{G}$ be a strongly connected Marked WEG. If $\mathcal{G}$ is expansible, then there exists a minimum vector $N^\star = (N_1^\star, \ldots, N_n^\star) \in (\mathbb{N} - \{0\})^{\star n}$ such that

$$\mathcal{S} = \{\lambda \cdot N^\star, \lambda \in \mathbb{N} - \{0\}\}.$$

The Marked Event graph associated with $N^\star$ is then the minimum expansion of $\mathcal{G}$.

PROOF

Let $N^\star = (N_1^\star, \dots, N_n^\star)$ be the element from $\mathcal{S}$ with $N_1^\star$ minimum and let $N = (N_1, \dots, N_n) \in \mathcal{S}$. For every place $p = (t_i, t_j)$, $\dfrac{N_i}{N_i^\star} = \dfrac{N_j}{N_j^\star}$. As $\mathcal{G}$ is connected, there exists two prime strictly positive integers a and b such that $\dfrac{N_1}{N_1^\star} = \dfrac{N_2}{N_2^\star} = \dots = \dfrac{N_n}{N_n^\star} = \dfrac{a}{b}$. Thus $N = \dfrac{a}{b} \cdot N^\star$.

By contradiction, let suppose that $b > 1$ then $\forall l \in \{1, \dots, n\}$, b is a divisor of $N_l^\star$. Thus, there exists an integer vector $k = (k_1, \dots, k_n)$ with $k_l = \dfrac{N_l^\star}{b}$. Thus $k \in \mathcal{S}$ with $k_1 < N_1^\star$, the contradiction.

Lastly, since elements from $\mathcal{S}$ are proportional, N has all its components minimum in $\mathcal{S}$. □

The system $\Sigma(\mathcal{G})$ associated with the Marked WEG pictured by Figure 1.3 is

$$\Sigma(\mathcal{G}) = \begin{cases} N_2/5 = N_3/2 \\ N_3/1 = N_4/3 \\ N_2/5 = N_4/6 \\ N_1/3 = N_2/1 \\ N_4/2 = N_1/5 \end{cases}$$

The minimum integer solution is then $N^\star = (15, 5, 2, 6)$.

1.4.4 Relationship between expansion and normalization

THEOREM 1.3

Let $\mathcal{G}$ be a strongly connected WEG. $\mathcal{G}$ is expansible iff $\mathcal{G}$ is normalizable. Moreover, there exists $K \in \mathbb{N} - \{0\}$ such that, for any $t_i \in T$, $Z_i \cdot N_i = K$.

PROOF

$A \Rightarrow B$ If $\mathcal{G}$ is expansible then there exists a vector $N = (N_1, \dots, N_n) \in (\mathbb{N} - \{0\})^n$ such that for any place $p = (t_i, t_j)$, $\dfrac{N_i}{v(p)} = \dfrac{N_j}{u(p)}$. Let us define M as the least common multiple of integers $N_1, \dots, N_n$, $M = lcm_{t_i \in T} N_i$.

For every place $p = (t_i, t_j) \in P$, we set $\gamma_p = \dfrac{M}{N_i \cdot u(p)} = \dfrac{M}{N_j \cdot v(p)}$.

Then, for any couple of places $(p_a, p_b) \in \mathcal{P}^+(t_i) \times \mathcal{P}^-(t_i)$, $\gamma_{p_a} = \dfrac{M}{N_i \cdot u(p_a)}$

and $\gamma_{p_b} = \dfrac{M}{N_i \cdot v(p_b)}$. So,

$$\gamma_{p_a} \cdot u(p_a) = \frac{M}{N_i} = \gamma_{p_b} \cdot v(p_b)$$

and setting $Z_i = \dfrac{M}{N_i}$ for any $t_i \in T$, we get that $\mathcal{G}$ can be normalized.

$B \Rightarrow A$ Conversely, let us assume now that $\mathcal{G}$ is normalized. So, for every place $p = (t_i, t_j) \in P$, $v(p) = Z_j$ and $u(p) = Z_i$. Let us define $M = lcm_{t_i \in T} Z_i$ and $\forall t_i \in T$, $N_i = \dfrac{M}{Z_i}$. Then, for any place $p = (t_i, t_j) \in P$,

$$\frac{N_i}{v(p)} = \frac{N_i}{Z_j} = \frac{M}{Z_i \cdot Z_j} = \frac{N_j}{Z_i} = \frac{N_j}{u(p)},$$

and $\mathcal{G}$ is expansible.

Now, if $\mathcal{G}$ is normalized, we get $\dfrac{N_j}{Z_i} = \dfrac{N_i}{Z_j}$ for every place $p = (t_i, t_j) \in P$. Since $\mathcal{G}$ is strongly connected, it exists an integer $K > 0$ such that, for every $t_i \in T$, $Z_i \cdot N_i = K$. □

For the example pictured by Figure 1.3, we get

$$Z_1 \cdot N_1^\star = Z_2 \cdot N_2^\star = Z_3 \cdot N_3^\star = Z_4 \cdot N_4^\star = 30.$$

1.4.4.1 Liveness and maximal throughput of a unitary Timed Marked WEG using its minimum expansion

The main interest of the expansion is to get an algorithm to check the liveness and to compute the optimal throughput of a unitary Timed Marked WEG. Indeed, the expansion is a (usual) Timed Marked Event Graph. As noticed in Section 1.2.2, it corresponds then to usual uniform constraints, for which there exists polynomial algorithms for the these two problems (*see* Chapter **??**).

The main drawback of this method is that the minimum number of vertices of an expansion may be exponential. Thus, computing the expansion may not be possible for a wide class of graphs.

As example, let us consider the circuit of n transition pictured by Figure 1.6. The numbers of duplicates of an expansion verify $N_n = \dfrac{N_1}{2^{n-1}}$ and for every $i \in \{1, \dots, n-1\}$, $N_{i+1} = \dfrac{N_i}{2}$. A minimum integer solution is then, for every $i \in \{1, \dots, n\}$, $N_i^\star = 2^{n-i}$ and the size of the minimum expansion is in $\mathcal{O}(2^n)$. Thus, its size might be exponential and using the expansion might not be suitable for a wide class of graphs.

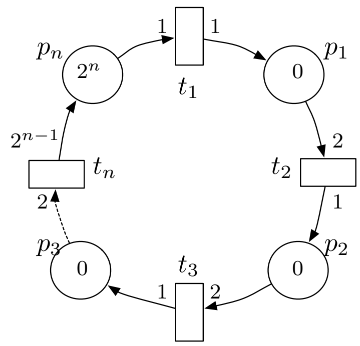

FIGURE 1.6: The number of vertices of the minimum expansion of $\mathcal{G}_n$ is exponential.

1.5 Periodic schedule of a Normalized Marked Timed WEG

This section is dedicated to the presentation of polynomial algorithms to check the existence and to compute the optimal throughput of a periodic schedule for a Normalized Marked Timed WEG. These results were first presented in [3]. From a practical point of view, the limitation to periodic schedules is often considered by many authors (*see* [**?**] to get schedules easier to implement.

The complexity of these two problems is unknown for general (non periodic) cyclic schedules. So polynomial sufficient condition of liveness and an upper bound of the optimum throughput can be immediately derived from the two algorithms presented here.

Periodic schedules are first formally defined. Then, for every place $p = (t_i, t_j)$, a condition on the starting time of the first execution t_i and t_j is expressed to fulfill the precedence relations induced by P. A polynomial algorithm checking the existence and computing a periodic schedule is then deduced. A simple example is lastly presented to illustrate that the throughput of a periodic schedule may be quite far from the optimal throughput.

1.5.1 Periodic schedules

DEFINITION 1.7 *A schedule σ is periodic if each transition $t_i \in T$ has a period w_i^σ such that*

$$\forall k \geq 0, S^\sigma_{<t_i,k>} = s_i^\sigma + (k-1) \cdot w_i^\sigma.$$

s_i^σ is the starting time of the first firing of t_i. The other firings of t_i are then repeated every w_i^σ time units.

1.5.2 Properties of periodic schedule

LEMMA 1.8
Let us consider a place $p = (t_i, t_j) \in P$, and let the integer values $k_{min} = \dfrac{\max(u(p) - v(p), 0) - M_0(p)}{gcd_p}$ and $k_{max} = \dfrac{u(p) - M_0(p)}{gcd_p} - 1$.

1. *If p induces a precedence relation between the firings $< t_i, \nu_i >$ and $< t_j, \nu_j >$ then there exists $k \in \{k_{min}, \dots, k_{max}\}$ such that $u(p) \cdot \nu_i - v(p) \cdot \nu_j = k \cdot gcd_p$.*
2. *Conversely, for any $k \in \{k_{min}, \dots, k_{max}\}$, there exists an infinite number of tuples $(\nu_i, \nu_j) \in (\mathbb{N} - \{0\})^2$ such that $u(p) \cdot \nu_i - v(p) \cdot \nu_j = k \cdot gcd_p$ and p induces a precedence relation between firings $< t_i, \nu_i >$ and $< t_j, \nu_j >$.*

PROOF

1. Since $gcd_p = gcd(v(p), u(p))$, for any tuple $(\nu_i, \nu_j) \in (\mathbb{N} - \{0\})^2$ there exists $k \in \mathbb{Z}$ such that $u(p) \cdot \nu_i - v(p) \cdot \nu_j = k \cdot gcd_p$. Now, if there is a precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$, we get by Lemma 1.1, assuming by Lemma 1.3 that $M_0(p)$ is a multiple of gcd_p,

   $$u(p) - M_0(p) > u(p) \cdot \nu_i - v(p) \cdot \nu_j \geq \max(u(p) - v(p), 0) - M_0(p),$$

   which is equivalent to

   $$u(p) - M_0(p) - gcd_p \geq k \cdot gcd_p \geq \max(u(p) - v(p), 0) - M_0(p).$$

   So we get $k_{min} \leq k \leq k_{max}$.

2. Conversely, there exists $(a, b) \in \mathbb{Z}^2$ such that $a \cdot u(p) - b \cdot v(p) = gcd_p$. Then for any $k \in \{k_{min}, \dots, k_{max}\}$, and any integer $q \geq 0$, the couple of integers $(\nu_i, \nu_j) = (k \cdot a + q \cdot v(p), k \cdot b + q \cdot u(p))$ is such that $u(p) \cdot \nu_i - v(p) \cdot \nu_j = k \cdot gcd_p$. Thus p induces a precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$, which achieves the proof.

$\square$

THEOREM 1.4
Let $\mathcal{G}$ be a Normalized Timed Marked WEG. For any periodic schedule σ, there exists a rational $K^\sigma \in \mathbb{Q}^+ - \{0\}$ such that, for any couple of transitions $(t_i, t_j) \in T^2$, $\dfrac{w_i^\sigma}{Z_i} = \dfrac{w_j^\sigma}{Z_j} = K^\sigma$. Moreover, the precedence relations associated with any place $p = (t_i, t_j)$ are fulfilled by σ iff

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + K^\sigma \cdot (Z_j - M_0(p) - gcd_p).$$

PROOF Let be a place $p = (t_i, t_j) \in P$ inducing a precedence relation between the firings $< t_i, \nu_i >$ and $< t_j, \nu_j >$. Then,

$$S^\sigma_{<t_i,\nu_i>} + \ell(t_i) \leq S^\sigma_{<t_j,\nu_j>}.$$

Since σ is periodic, we get

$$s_i^\sigma + (\nu_i - 1) \cdot w_i^\sigma + \ell(t_i) \leq s_j^\sigma + (\nu_j - 1) \cdot w_j^\sigma.$$

Then, by Lemma 1.8, there exists $k \in \{k_{min}, \ldots, k_{max}\}$ such that $\nu_j = \dfrac{u(p) \cdot \nu_i - k \cdot gcd_p}{v(p)}$ and

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + w_j^\sigma - w_i^\sigma + \nu_i \cdot w_i^\sigma - \frac{u(p) \cdot \nu_i - k \cdot gcd_p}{v(p)} \cdot w_j^\sigma.$$

So, $s_j^\sigma - s_i^\sigma \geq \ell(t_i) + \left(w_i^\sigma - \dfrac{u(p)}{v(p)} \cdot w_j^\sigma\right) \cdot \nu_i + \left(1 + \dfrac{k \cdot gcd_p}{v(p)}\right) \cdot w_j^\sigma - w_i^\sigma$. This inequality must be true for any value $\nu_i \in \mathbb{N} - \{0\}$, so $w_i^\sigma - \dfrac{u(p)}{v(p)} \cdot w_j \leq 0$ and then $\dfrac{w_i^\sigma}{u(p)} \leq \dfrac{w_j^\sigma}{v(p)}$. As $\mathcal{G}$ is normalized, $u(p) = Z_i$ and $v(p) = Z_j$. Since $\mathcal{G}$ is unitary, it is strongly connected and thus, for any place $p = (t_i, t_j)$, $\dfrac{w_i^\sigma}{Z_i} = \dfrac{w_j^\sigma}{Z_j}$. So, there exists a value $K^\sigma \in \mathbb{Q} - \{0\}$ such that, for any transition $t_i \in T$, $\dfrac{w_i^\sigma}{Z_i} = K^\sigma$. Then, the previous inequality becomes

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + K^\sigma \cdot Z_j \cdot \left(1 + \frac{k \cdot gcd_p}{Z_j}\right) - K^\sigma \cdot Z_i$$

and thus

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + K^\sigma \cdot (Z_j - Z_i + k \cdot gcd_p).$$

Now, the right term grows with k and according to Lemma 1.8 there exists $(\nu_i, \nu_j) \in (\mathbb{N} - \{0\})^2$ such that $k = k_{max}$, thus the precedence relation holds iff

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + K^\sigma \cdot (Z_j - Z_i + Z_i - M_0(p) - gcd_p)$$

which is equivalent to

$$s_j^\sigma - s_i^\sigma \geq \ell(t_i) + K^\sigma \cdot (Z_j - M_0(p) - gcd_p).$$

Conversely, assume this last inequality and that $\forall t_i \in T$, $\dfrac{w_i^\sigma}{Z_i} = K^\sigma$. Then, for any integers ν_i and ν_j with $u(p) \cdot \nu_i - v(p) \cdot \nu_j = k \cdot gcd_p$ for $k \in \{k_{min}, \ldots, k_{max}\}$, we can prove that σ checks the precedence relation between $< t_i, \nu_i >$ and $< t_j, \nu_j >$. □

1.5.3 Existence of periodic schedules

The constraints expressed by Theorem 1.4 may be modelled by a valued graph $G = (X, A)$ built as follows:

1. the set of vertices is the set of transitions, *i.e.* $X = T$;

2. To each place $p = (t_i, t_j)$ is associated an arc $a = (t_i, t_j)$ valued by $v(a, K^\sigma) = \ell(t_i) + K^\sigma \cdot (Z_j - M_0(p) - gcd_p)$. Following the notation of Chapter **??**, we set $L(a) = \ell(t_i)$, $H(a) = M_0(p) + gcd_p - Z_j$ to obtain $v(a, K^\sigma) = L(a) - K^\sigma \cdot H(a)$.

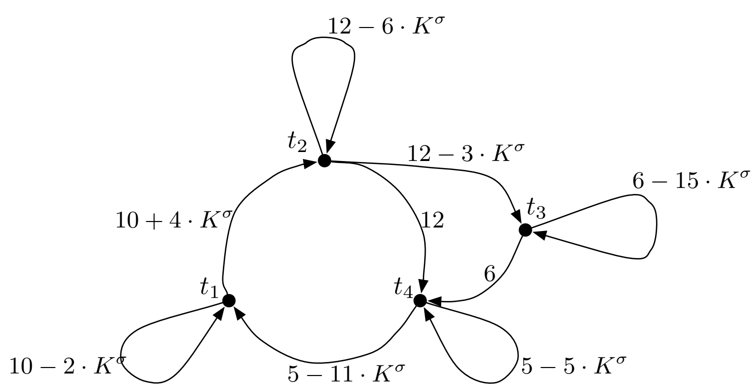

FIGURE 1.7: Valued graph $G = (X, A)$ associated with the normalized Marked WEG pictured by Figure 1.4.

For a given value $K^\sigma \in \mathbb{Q}^+$, the set of inequalities on the starting times of the first firings of the transitions is a difference constraints system as defined by Lemma **??** in Chapter **??**. By extension, for every path μ of G, we set

$$L(\mu) = \sum_{a \in \mu} L(a) \text{ and } H(\mu) = \sum_{a \in \mu} H(a).$$

Since $L(a) > 0$ for every arc $a \in A$, the following theorem is easily deduced from Lemma **??** in Chapter **??**:

THEOREM 1.5
Let $\mathcal{G}$ be a Normalized Timed WEG. There exists a periodic schedule iff, for every circuit c of G, $H(c) > 0$.

Surprisingly, this condition is similar to a sufficient condition of liveness proved in [23]. An algorithm in $\mathcal{O}(\max(nm, m \max_{t_i \in T} \log Z_i))$ to evaluate this condition can be found in this paper. It is also proved in [23] that this condition is a necessary and sufficient condition of liveness for circuits composed by two transitions. So, the following corollary is easily deduced:

COROLLARY 1.1
Let $\mathcal{G}$ be a Normalized Marked Timed WEG composed by a circuit of two transitions. $\mathcal{G}$ is live iff $\mathcal{G}$ has a periodic schedule.

This corollary is not true anymore for circuits with 3 transitions. For example, let us consider the Normalized Timed WEG $\mathcal{G}$ presented by Figure 1.8 with no particular assumption on firing durations. The sequence of firings $s = t_3t_1t_1t_1t_2t_3t_1t_1t_1t_1t_2t_2$ can be repeated infinitely, so it is live.

However, for the circuit $c = t_1t_2t_3t_1$ we get:

$$H(c) = \sum_{i=1}^{3} M_0(p_i) + \sum_{i=1}^{3} gcd_{p_i} - \sum_{i=1}^{3} Z_i = 28 + 12 - 41 < 0$$

so the condition of Theorem 1.5 is false and this circuit has no periodic schedule.

1.5.4 Optimal periodic schedule

Assume here that $\mathcal{G}$ is a Normalized Timed WEG which fulfills the condition expressed by Theorem 1.5. Then, the minimum value of K^σ is

$$K^{opt} = \max_{c \in C(G)} \frac{L(c)}{H(c)}$$

where $C(G)$ is the set of the circuits of G. K^{opt} is the minimum cycle mean of G, as defined by [17]. The computation of K^{opt} and the determination of a corresponding constraint graph was discussed in Chapter **??**.

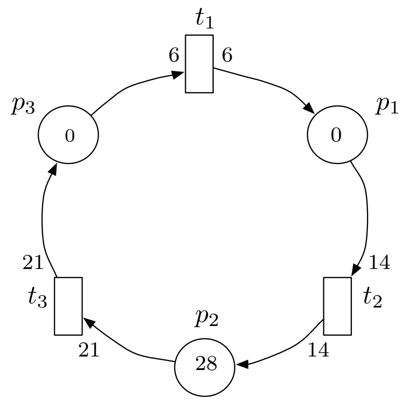

FIGURE 1.8: $\mathcal{G}$ is live but has no periodic schedule.

Now, we can observe that the throughput of a periodic schedule may be quite far from the optimum. For example, let us consider a Marked Normalized Timed WEG circuit composed by two places $p_1 = (t_1, t_2), p_2 = (t_2, t_1)$ such that $gcd_{p_1} = gcd_{p_2} = 1$, $M_0(p_1) = Z_2 + Z_1 - 1$ and $M_0(p_2) = 0$ (*see* Fgure 1.9).

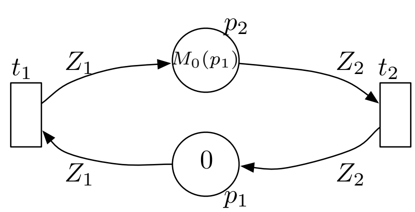

FIGURE 1.9: A unitary WEG with two places with $gcd_{p_1} = gcd_{p_2} = 1$ and $M_0(p_1) = Z_2 + Z_1 - 1$.

It fulfills the condition stated by Theorem 1.5:

$$M_0(p_1) + M_0(p_2) + gcd_{p_1} + gcd_{p_2} - Z_2 - Z_1 = 1 > 0$$

The associated bi-valued graph G is then pictured by Figure 1.10.

We get $K^{opt} = \max\left\{\frac{\ell(t_1)}{Z_1}, \frac{\ell(t_2)}{Z_2}, \ell(t_1) + \ell(t_2)\right\} = \ell(t_1) + \ell(t_2)$ and the throughput of transitions for the optimum periodic schedule σ is $\tau_1^\sigma = \dfrac{1}{w_1^\sigma} =$

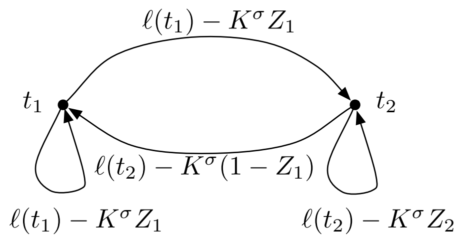

FIGURE 1.10: Bi-valued graph G associated with the normalized TWEG with two places.

$\dfrac{1}{Z_1 \cdot (\ell(t_1) + \ell(t_2))}$ and $\tau_2^\sigma = \dfrac{1}{w_2^\sigma} = \dfrac{1}{Z_2 \cdot (\ell(t_1) + \ell(t_2))}$.

Let us consider now the earliest schedule σ' of the latter example. Since the total number of tokens in the circuit is $Z_1 + Z_2 - 1$, transitions t_1 and t_2 will never be fired simultaneously by σ'. Moreover, if we denote by n_1 (*resp.* n_2) the number of firings of t_1 (*resp.* t_2) such that the system will return to its initial state (*i.e.* with Z_1+Z_2-1 tokens in p_1 and 0 token in p_2), then we must have $n_1 \cdot Z_1 - n_2 \cdot Z_2 = 0$. Thus, there exists $k \in \mathbb{N} - \{0\}$ with $n_1 = k \cdot Z_2$ and $n_2 = k \cdot Z_1$. The throughput of transitions t_1 and t_2 for the earliest schedule is then $\tau_1^{\sigma'} = \dfrac{Z_2}{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2)}$ and $\tau_2^{\sigma'} = \dfrac{Z_1}{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2)}$. Let us define now the ratio

$$R = \frac{\tau_1^{\sigma'}}{\tau_1^\sigma} = \frac{\tau_2^{\sigma'}}{\tau_2^\sigma} = \frac{Z_1 \cdot Z_2 \cdot (\ell(t_1) + \ell(t_2))}{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2)}.$$

Assume without loss of generality that $Z_1 \geq Z_2$, then

$$R = Z_1 \cdot \left(\frac{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2) - (Z_1 - Z_2) \cdot \ell(t_2)}{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2)} \right).$$

So,

$$R = Z_1 \cdot \left(1 - \frac{(Z_1 - Z_2) \cdot \ell(t_2)}{Z_2 \cdot \ell(t_1) + Z_1 \cdot \ell(t_2)} \right) < Z_1.$$

The ratio R is then maximum when $\ell(t_1)$ tends to infinity and the bound $\max(Z_1, Z_2)$ is asymptotically reached.

1.6 Conclusion

This chapter presented some basic recent advances on Timed Weighted Event Graphs. It was focused on two open questions on these systems, namely the development of efficient algorithms for checking the liveness and computing the maximal throughput. They are fundamental for a practical point of view since most of the optimization problems expressed on these systems needed need to solve them efficiently in order to evaluate the solutions obtained.

As we noticed in this chapter, the complexity of these two previous problem is still open and is also a challenging theoretical question. If no polynomial algorithm exists, the computation of another lower bound for the maximum throughput should also be investigated to improve these presented here.

The mathematical tools presented here also allow to solve polynomially two optimization problems: the computation of an initial live marking minimizing the places capacities is developped in [22] based on the sufficient condition of liveness expressed by Theorem 1.5. An approximation algorithm was also developed in [24] which maximizes the throughput for place capacities at most twice from the minimum. These two algorithms illustrates that, efficient polynomial algorithms may be obtained for some particular problems on WEG, even if the general problem does not necessarily belong to $\mathcal{NP}$.

1.7 Acknowledgments

We wish to thank Pascal Urard from STMicroelectronics for its encouragement and its interest on our work. We are also very grateful to Claire Hanen and Mohammed Benazzouz for their numerous constructive remarks on this paper.

References

[1] M. Adé. *Data memory minimization for synchronous data flow graphs emulated on DSP-FPGA targets.* PhD thesis, Université Catholique de Louvain, 1997.

[2] M. Adé, R. Lauwereins, and J. A. Peperstraete. Buffer memory requirements in dsp applications. In *IEEE 5th International Workshop on Rapid System Prototyping*, pages 108–123, Grenoble, France, 1994.

[3] Abir Benabid, Claire Hanen, Olivier Marchetti, and Alix Munier-Kordon. Periodic schedules for unitary timed weighted event graphs. In *ROADEF 2008 conference*, Clermont-Ferrand, France, 2008.

[4] Shuvra S. Bhattacharyya, Praveen K. Murthy, and Edward A. Lee. *Software Synthesis from dataflow graphs*. Kuwer, first edition, 1996.

[5] Shuvra S. Bhattacharyya, Praveen K. Murthy, and Edward A. Lee. Synthesis of embedded software from synchronous dataflow specifications. *Journal of VLSI Signal Processing*, (21):151–166, 1999.

[6] P. Chrzastowski-Wachtel and M. Raczunas. Liveness of weighted circuits and the diophantine problem of frobenius. In *Proceeding of FCT'93, Lecture Notes in Computer Science*, volume 710, pages 171–180. Springer, 1993.

[7] Thomas Cormen, Charles Leiserson, and Ronald Rivest. *Introduction to Algorithms*. MIT Press, 1990.

[8] M. Čubrić and P. Panangaden. Minimal memory schedules for dataflow networks. In *CONCUR '93, 4th International Conference on Concurrency Theory, Lecture Notes in Computer Science*, volume 715, pages 368–383, 1993.

[9] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co, 1979.

[10] Stéphane Gaubert. An algebraic method for optimizing resources in timed event graphs. In *Proceedings of the 9th conference on Analysis and Optimization of Systems, Lecture Notes in Computer Science*, volume 144, pages 957–966. Springer, 1990.

[11] Marc Geilen, Twan Basten, and Sander Stuijk. Minimising buffer requirements of synchronous dataflow graphs with model cheking. In *Proocedings of the DAC*, 2005.

[12] A.H Ghamarian, M Geilen, T Basten, A.J.M Moonen, M.J.G Bekooij, B.D Theelen, and M.R Mousavi. Throughput analysis of synchronous data flow graphs. In *Proocedings of the Sixth IEEE International Conference on Applications of Concurrency for System Design*, 2006.

[13] A. Giua, A. Piccaluga, and C. Seatzu. Firing rate optimization of cyclic timed event graphs. *Automatica*, 38(1):91–103, January 2002.

[14] H. Hillion and J.M. Proth. Performance evaluation of a job-shop system using timed event graph. *IEEE transactions on automatic control*, 34(1):3–9, 1989.

[15] Michal Karczmarek. Constrained and phased scheduling of synchronous data flow graphs for streamit language, 2002.

[16] R. M. Karp and R. E. Miller. Properties of a model for parallel computations: Determinacy, termination, queueing. *SIAM*, 14(63):1390–1411, 1966.

[17] Richard M Karp. A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics*, 23, 1978.

[18] S. Laftit, J.M. Proth, and X. Xie. Optimization of invariant criteria for event graphs. *IEEE Transactions on Automatic Control*, 37(5):547–555, 1992.

[19] Edward A Lee. Overview of the ptolemy project. Technical report, University of California, Berkeley, USA, 2003.

[20] Edward A. Lee and David G. Messerschmitt. Static scheduling of synchronous data flow programs for digital signal processing. *IEEE Transaction on Computers*, 36(1), 1987.

[21] Edward A. Lee and David G. Messerschmitt. Synchronous data flow. *IEEE Proceedings of the IEEE*, 75(9), 1987.

[22] O. Marchetti and A. Munier-Kordon. Minimizing place capacities of weighed event graphs for enforcing liveness. *To appear in Discrete Event Dynamic Systems*, 2008.

[23] Olivier Marchetti and Alix Munier-Kordon. A sufficient condition for the liveness of weighted event graphs. *Submitted, LIP6 Research Report, http://ftp.lip6.fr/lip6/reports/2005/lip6-2005-004.pdf*, 2004.

[24] Olivier Marchetti and Alix Munier-Kordon. Complexity results for bi-criteria cyclic scheduling problems. *Submitted, LIP6 Research Report, http://www-asim.lip6.fr/publications/resumes/2006/rp.2006.marchetti2/ index.gb.html*, 2006.

[25] A. Munier. The basic cyclic scheduling problem with linear precedence constraints. *Discrete Applied Mathematics*, 64(3):219–238, 1996.

[26] Alix Munier. Régime asymptotique optimal d'un graphe d'événements temporisé généralisé: application à un problèmed'assemblage. *RAIRO-Automatique Productique Informatique Industrielle*, 27(5):487–513, 1993.

[27] P. M. Murthy. *Scheduling Techniques for Synchronous and Multidimensional Synchronous Dataflow.* PhD thesis, University of California at Berkeley, 1996.

[28] Praveen K. Murthy, Shuvra S. Bhattacharyya, and Edward A. Lee. Joint minimization of code and data for synchonous dataflow programs. *Journal of Formal Methods in System Design*, 11(1), 1997.

[29] J-L Peterson. *Petri Net Theory and the Modeling of Systems.* Prentice Hall PTR, 1981.

[30] Nathalie Sauer. Marking optimization of weighted marked graph. *Discrete Event Dynamic Systems*, 13(3):245–262, 2003.

[31] Sander Stuijk, Marc Geilen, and Twan Basten. Exploring trade-offs in buffer requirements and throughput constraints for synchronous dataflow graphs. In *DAC '06: Proceedings of the 43rd annual conference on Design automation*, pages 899–904, New York, NY, USA, 2006. ACM.

[32] E. Teruel, P. Chrzastowski-Wachtel, J. M. Colom, and M. Silva. On weighted T-systems. In *Proocedings of the 13th Internationnal Conference on Application and Theory of Petri Nets 1992, Lecture Notes in Computer Science*, volume 616. Springer, 1992.

[33] William Thies, Michal Karczmarek, Michael Gordon, David Z. Maze, Jeremy Wong, Henry Hoffman, Matthew Brown, and Saman Amarasinghe. Streamit: A compiler for streming applications. Technical report, Massachussetts institute of technology, 2001.